

Hamiltonian Lattice QCD with Wilson Fermions at Strong Coupling

Yi-Zhong Fang and Xiang-Qian Luo*

Department of Physics, Zhongshan University, Guangzhou 510275, People's Republic of China

August 22, 2001

Abstract

Hamiltonian lattice QCD with Wilson fermions is investigated systematically by strong-coupling expansion up to the second order. The effective Hamiltonian is diagonalized by Bogoliubov transformation. The vacuum energy, chiral condensate, pseudo-scalar and vector meson masses are calculated. The comparison with the unitary transformation method by Luo and Chen is also made. The method discussed in this paper has potential application to QCD at finite density.

*Corresponding author. E-mail: stslxq@zsu.edu.cn

1 Introduction

Lattice Gauge Theory (LGT) proposed by Wilson [1] is the most promising non-perturbative method for QCD. In comparison to other techniques, the advantage of LGT is that there is no free parameter when the continuum limit is taken. There are two different approaches: Lagrangian formulation and Hamiltonian formulation. Lagrangian formulation is convenient for Monte Carlo simulations. Hamiltonian formulation is useful for analytical calculations [2, 3, 4, 5, 6, 7]. Although the standard lattice Lagrangian Monte Carlo method has been successful in many aspects of LGT, it unfortunately breaks down at finite chemical potential due to the “complex action problem”. On the other hand, lattice QCD at finite chemical potential formulated in the Hamiltonian approach does not encounter the complex action problem. Recently, Gregory, Guo, Kröger and Luo [8] investigated the finite density Hamiltonian lattice QCD with naive fermions, using the unitary transformation, variational method and Bogoliubov transformation method developed by Luo and Chen [5].

It is well known that naive fermions experience “species doubling”. Kogut-Susskind fermions and Wilson fermions are the most popular ways towards solving of the problem. Although the doubling problem has not yet been completely solved, Kogut-Susskind fermions are frequently used for exploring the spontaneous breaking of the U(1) chiral symmetry. For Wilson fermions, the doubling modes are removed, but chiral symmetry is explicitly broken and fine tuning of the fermion mass remains to be done; This method is very popular in spectrum computations, due to the existence of the flavor symmetry.

In this paper, we study the Hamiltonian lattice QCD with Wilson fermions by strong coupling expansion method. The effective Hamiltonian is Fierz-rearranged, re-expressed by meson operators, and then diagonalized by Bogoliubov transformation. The vacuum energy, chiral condensate, pseudo-scalar and vector meson masses are computed.

The rest of the paper is organized as follows. In Sect. 2, we obtain the effective Hamiltonian using strong coupling expansion. The physical results are presented in Sect. 3 and Sect. 4. The conclusion is provided in Sect. 5.

2 Effective Hamiltonian at strong coupling

We begin with the (d+1)-dimensional lattice QCD Hamiltonian with Wilson fermions

$$\begin{aligned}
H &= \left(m + \frac{rd}{a}\right) \sum_x \bar{\psi}(x)\psi(x) + \frac{1}{2a} \sum_x \sum_{k=\pm 1}^{\pm d} \bar{\psi}(x)\gamma_k U(x, k)\psi(x+k) \\
&- \frac{r}{2a} \sum_x \sum_{k=\pm 1}^{\pm d} \bar{\psi}(x)U(x, k)\psi(x+k) + \frac{g^2}{2a} \sum_y \sum_{j=1}^d E_j^\alpha(x)E_j^\alpha(x) \\
&- \frac{1}{ag^2} \sum_p \text{Tr} \left(U_p + U_p^\dagger - 2 \right), \tag{1}
\end{aligned}$$

where m , a , r and g are respectively the bare fermion mass, lattice spacing, Wilson parameter, and bare coupling constant. $U(x, k)$ is the gauge link variable at site x and direction k , and ψ is the four-component spinor. The convention $\gamma_{-k} = -\gamma_k$ is used. $E^\alpha(x)$ is the color-electric field and summation over $\alpha = 1, 2, \dots, 8$ is implied. U_p is the product of gauge link variable around an elementary plaquette and in the continuum, it represents the color magnetic interactions. However, at the strong coupling, the color magnetic energy (the last term) can be ignored.

We want to diagonalize H so that the fermion field ψ can be expressed in terms of up and down 2-spinors ξ and η^\dagger

$$\psi(x) = \begin{pmatrix} \xi(x) \\ \eta^\dagger(x) \end{pmatrix}. \quad (2)$$

The bare vacuum state $|0\rangle$ is defined by $\xi|0\rangle = \eta|0\rangle = E_j^\alpha(x)|0\rangle = 0$. Since the up and down components are coupled via the γ_k matrices, the bare vacuum is not an eigenstate of H .

For the convenience of strong coupling expansion, we introduce the un-perturbative term W_0 and perturbative term W , i.e.,

$$\frac{2a}{g^2}H = W_0 + W, \quad (3)$$

where

$$\begin{aligned} W_0 &= (m + \frac{rd}{a})\frac{2a}{g^2} \sum_x \bar{\psi}(x)\psi(x) + \sum_{x,j} E_j^\alpha(x)E_j^\alpha(x), \\ W &= \frac{1}{g^2} \sum_{x,k} \bar{\psi}(x)(-r + \gamma_k)U(x, k)\psi(x + k). \end{aligned} \quad (4)$$

The ground state energy is $g^2\varepsilon/2a$, where for $1/g^2 \ll 1$, ε can be approximated by

$$\varepsilon = \langle 0 | \frac{2a}{g^2}H | 0 \rangle = \varepsilon^{(0)} + \varepsilon^{(1)} + \varepsilon^{(2)}. \quad (5)$$

$\varepsilon^{(0)}$, $\varepsilon^{(1)}$ and $\varepsilon^{(2)}$ are the zeroth, first and second order corrections:

$$\begin{aligned} \varepsilon^{(0)} &= W_0|0\rangle, \\ \varepsilon^{(1)} &= \langle 0 | W | 0 \rangle, \\ \varepsilon^{(2)} &= \langle 0 | W \frac{1 - |0\rangle\langle 0|}{\varepsilon^{(0)} - W_0} W | 0 \rangle. \end{aligned} \quad (6)$$

From (4) and (6), we obtain

$$\begin{aligned} \varepsilon^{(1)} &= 0, \\ \varepsilon^{(0)} &= \frac{2aM}{g^2} \langle 0 | \sum_x \bar{\psi}(x)\psi(x) | 0 \rangle = -2N_c N_f N_s \frac{2aM}{g^2}. \end{aligned} \quad (7)$$

Γ_A	1	$i\gamma_j$	γ_4	$i\gamma_5$	$\gamma_4\gamma_5$	$\gamma_4\gamma_j$	$i\epsilon_{jj_1j_2}\gamma_{j_1}\gamma_{j_2}$	$i\epsilon_{jj_1j_2}\gamma_4\gamma_{j_1}\gamma_{j_2}$
L'_A	1	1	1	1	1	1	1	1

Table 1: Γ matrices and coefficients for H_r^{eff} .

Here N_s , N_c and N_f are respectively the number of lattice sites, colors and flavors. M

$$M = m + \frac{rd}{a}, \quad (8)$$

and $1/(2Ma)$ is an analog of the κ parameter in Lagrangian LGT. Using

$$\begin{aligned} W_0 W|0\rangle &= [W_0, W]|0\rangle + W W_0|0\rangle \\ &= [W_0, W]|0\rangle + \varepsilon^{(0)} W|0\rangle, \end{aligned} \quad (9)$$

the second order correction becomes

$$\varepsilon^{(2)} = \langle 0|W \frac{1 - |0\rangle\langle 0|}{\varepsilon^{(0)} - W_0} W|0\rangle = -\frac{1}{C_N} \langle 0|W (1 - |0\rangle\langle 0|) W|0\rangle = -\frac{1}{C_N} \langle 0|WW|0\rangle, \quad (10)$$

where $C_N = \text{Tr} \sum_\alpha \lambda^\alpha \lambda^\alpha / N_C = (N_C^2 - 1)/(2N_C)$, i.e. the Casimir invariant. From (7) and (10), we have

$$\varepsilon = \varepsilon^{(0)} - \langle 0|\frac{WW}{C_N}|0\rangle = \langle 0|\frac{2a}{g^2} H_{eff}|0\rangle. \quad (11)$$

Therefore, the effective Hamiltonian is

$$H_{eff} = M \sum_x \bar{\psi}(x) \psi(x) + H_r^{eff} + H_k^{eff}. \quad (12)$$

This equation will serve as our starting point, where

$$H_r^{eff} = -\frac{Kr^2}{2aN_C} \sum_{x,k} \bar{\psi}_{c_1,f_1}(x) \psi_{c_2,f_1}(x+k) \bar{\psi}_{c_2,f_2}(x+k) \psi_{c_1,f_2}(x), \quad (13)$$

and

$$H_k^{eff} = \frac{K}{2aN_C} \sum_{x,k} \bar{\psi}_{c_1,f_1}(x) \gamma_k \psi_{c_2,f_1}(x+k) \bar{\psi}_{c_2,f_2}(x+k) \gamma_k \psi_{c_1,f_2}(x). \quad (14)$$

$K = 1/(g^2 C_N)$ is the effective coupling of the four-fermion interactions. The color index c and flavor index f are explicitly written, and summation over repeated index is implied. It is easily proven that the contribution from terms proportional to $O(r)$ vanishes.

Γ_A	1	$i\gamma_j$	γ_4	$i\gamma_5$	$\gamma_4\gamma_5$	$\gamma_4\gamma_j$	$i\epsilon_{jj_1j_2}\gamma_{j_1}\gamma_{j_2}$	$i\epsilon_{jj_1j_2}\gamma_4\gamma_{j_1}\gamma_{j_2}$
L_A	-1	$1 - 2\delta_{k,j}$	1	1	-1	$-1 + 2\delta_{k,j}$	$1 - 2\delta_{k,j}$	$-1 + 2\delta_{k,j}$

Table 2: Γ matrices and coefficients for H_k^{eff} .

In order to express the effective Hamiltonian by meson operators, we perform a Fierz transformation so that (13) and (14) are rearranged as,

$$H_r^{eff} = -\frac{Kr^2d}{a} \sum_x \psi^\dagger(x)\psi(x) + \frac{Kr^2}{8aN_c} \sum_{x,k} L'_A \psi_{f_1}^\dagger(x) \Gamma_A \psi_{f_2}(x) \psi_{f_2}^\dagger(x+k) \Gamma_A \psi_{f_1}(x+k), \quad (15)$$

and

$$H_k^{eff} = -\frac{Kd}{a} \sum_x \psi^\dagger(x)\psi(x) - \frac{K}{8aN_c} \sum_{x,k} L_A \psi_{f_1}^\dagger(x) \Gamma_A \psi_{f_2}(x) \psi_{f_2}^\dagger(x+k) \Gamma_A \psi_{f_1}(x+k). \quad (16)$$

where the matrices Γ_A and their coefficients L'_A and L_A are given in Tab. [1] and Tab.[2], and summation over the index A is understood. Note our basis of the sixteen Γ_A matrices is the same as the standard reference [11], but it is different from Smit[3].

Substituting (15) and (16) into (12), we have

$$\begin{aligned} H_{eff} = & M \sum_x \bar{\psi}(x)\psi(x) - \frac{K}{a}(r^2 + 1)d \sum_x \psi^\dagger(x)\psi(x) \\ & + \frac{K}{8aN_c} \sum_x \sum_{k=\pm j} \left[(r^2 + 1) \psi^\dagger(x)\psi(x) \psi^\dagger(x+k)\psi(x+k) \right. \\ & + (r^2 - 1) \psi^\dagger(x) \gamma_4 \psi(x) \psi^\dagger(x+k) \gamma_4 \psi(x+k) \\ & - (r^2 - 1) \psi^\dagger(x) \gamma_5 \psi(x) \psi^\dagger(x+k) \gamma_5 \psi(x+k) \\ & + (r^2 + 1) \psi^\dagger(x) \gamma_4 \gamma_5 \psi(x) \psi^\dagger(x+k) \gamma_4 \gamma_5 \psi(x+k) \\ & + \left(r^2 + (1 - 2\delta_{k,j}) \right) \psi^\dagger(x) \gamma_4 \gamma_j \psi(x) \psi^\dagger(x+k) \gamma_4 \gamma_j \psi(x+k) \\ & - \left(r^2 - (1 - 2\delta_{k,j}) \right) \psi^\dagger(x) \gamma_j \psi(x) \psi^\dagger(x+k) \gamma_j \psi(x+k) \\ & - \left((1 - 2\delta_{k,j}) + r^2 \right) \psi^\dagger(x) \gamma_4 \sigma_j \psi(x) \psi^\dagger(x+k) \gamma_4 \sigma_j \psi(x+k) \\ & \left. + \left((1 - 2\delta_{k,j}) - r^2 \right) \psi^\dagger(x) \sigma_j \psi(x) \psi^\dagger(x+k) \sigma_j \psi(x+k) \right], \quad (17) \end{aligned}$$

where $\sigma_j = \epsilon_{jj_1j_2} \gamma_{j_1} \gamma_{j_2}$. This effective Hamiltonian describes the nearest-neighbor four fermion interactions. For $r \rightarrow 0$, H_{eff} is the same as that from the unitary transformation method using by Luo and Chen[5].

Define the pseudo-scalar meson operator Π and the vector meson operator V as [8]

$$\Pi(x)_{f_1 f_2} = \frac{1}{2\sqrt{2N_c}} \psi_{f_1}^\dagger(x) (1 - \gamma_4) \gamma_5 \psi_{f_2}(x),$$

$$\begin{aligned}
\Pi^\dagger(x)_{f_2 f_1} &= \frac{1}{2\sqrt{2N_c}} \psi_{f_2}^\dagger(x) (1 + \gamma_4) \gamma_5 \psi_{f_1}(x), \\
V_j(x)_{f_1 f_2} &= \frac{1}{2\sqrt{2N_c}} \psi_{f_1}^\dagger(x) (1 - \gamma_4) \gamma_j \psi_{f_2}(x), \\
V_j^\dagger(x)_{f_2 f_1} &= \frac{1}{2\sqrt{2N_c}} \psi_{f_2}^\dagger(x) (1 + \gamma_4) \gamma_j \psi_{f_1}(x).
\end{aligned} \tag{18}$$

Then Eq. (17) can be re-expressed as

$$\begin{aligned}
H_{eff} &= (-2N_c N_f V) \left(M + \frac{Kd}{a} \right) + 2M \sum_{x, f_1 f_2 j} \left(\Pi_{f_2 f_1}^\dagger(x) \Pi_{f_1 f_2}(x) + V_{j f_2 f_1}^\dagger(x) V_{j f_1 f_2}(x) \right) \\
&- \frac{2Kr^2 d}{a} \sum_{x f_1 f_2 j} \left(\Pi_{f_2 f_1}^\dagger(x) \Pi_{f_1 f_2}(x) + V_{j f_2 f_1}^\dagger(x) V_{j f_1 f_2}(x) \right) \\
&- \frac{Kr^2}{2a} \sum_{x f_1 f_2 k j} \left(\Pi_{f_1 f_2}^\dagger(x) \Pi_{f_2 f_1}(x+k) + \Pi_{f_1 f_2}(x) \Pi_{f_2 f_1}^\dagger(x+k) \right. \\
&+ \left. V_{j f_1 f_2}^\dagger(x) V_{j f_2 f_1}(x+k) + V_{j f_1 f_2}(x) V_{j f_2 f_1}^\dagger(x+k) \right) \\
&+ \frac{K}{2a} \sum_{x, k} \left(\Pi_{f_1 f_2}^\dagger(x) \Pi_{f_2 f_1}^\dagger(x+k) + \Pi_{f_1 f_2}(x) \Pi_{f_2 f_1}(x+k) \right. \\
&+ \left. \left(V_{j f_1 f_2}^\dagger(x) V_{j f_2 f_1}^\dagger(x+k) + V_{j f_1 f_2}(x) V_{j f_2 f_1}(x+k) \right) (1 - 2\delta_{k,j}) \right) \\
&+ \frac{2Kd}{a} \left(\sum_{x f_1 f_2} \Pi_{f_2 f_1}^\dagger(x) \Pi_{f_1 f_2}(x) + \sum_{x f_1 f_2 j} V_{j f_2 f_1}^\dagger(x) V_{j f_1 f_2}(x) \right) \\
&= (-2N_c N_f N_s) \left(M + \frac{Kd}{a} \right) + H_\Pi + H_V,
\end{aligned} \tag{19}$$

where the contributions of the pseudo-scalar mesons are vector mesons are respectively

$$\begin{aligned}
H_\Pi &= \left(-\frac{2Kr^2 d}{a} + 2M \right) \sum_{x f_1 f_2} \Pi_{f_2 f_1}^\dagger(x) \Pi_{f_1 f_2}(x) \\
&- \frac{Kr^2}{2a} \sum_{x f_1 f_2 k} \left(\Pi_{f_1 f_2}^\dagger(x) \Pi_{f_2 f_1}(x+k) + \Pi_{f_1 f_2}(x) \Pi_{f_2 f_1}^\dagger(x+k) \right) \\
&+ \frac{K}{2a} \sum_{x f_1 f_2} \left(\Pi_{f_1 f_2}^\dagger(x) \Pi_{f_2 f_1}^\dagger(x+k) + \Pi_{f_1 f_2}(x) \Pi_{f_2 f_1}(x+k) \right) \\
&+ \frac{2Kd}{a} \sum_{x f_1 f_2} \Pi_{f_2 f_1}^\dagger(x) \Pi_{f_1 f_2}(x),
\end{aligned} \tag{20}$$

and

$$H_V = \left(2M - \frac{2Kr^2 d}{a} + \frac{2Kd}{a} \right) \sum_{x f_1 f_2 j} V_{j f_2 f_1}^\dagger(x) V_{j f_1 f_2}(x)$$

$$\begin{aligned}
& - \frac{Kr^2}{2a} \sum_{x f_1 f_2 k j} \left(V_{j f_1 f_2}^\dagger(x) V_{j f_2 f_1}(x+k) + V_{j f_1 f_2}(x) V_{j f_2 f_1}^\dagger(x+k) \right) \\
& + \frac{K}{2a} \sum_{x f_1 f_2 k j} \left(V_{j f_1 f_2}^\dagger(x) V_{j f_2 f_1}^\dagger(x+k) + V_{j f_1 f_2}(x) V_{j f_2 f_1}(x+k) \right) (1 - 2\delta_{k,j}). \quad (21)
\end{aligned}$$

3 Meson masses

After a Fourier transformation

$$\Pi_{f_1 f_2}(x) = \sum_p e^{ipx} a_{f_1 f_2}(p), \quad (22)$$

H_Π in (20) becomes

$$\begin{aligned}
H_\Pi &= \left(\frac{2Kd}{a}(1-r^2) + 2M \right) \sum_{p f_1 f_2} a_{f_1 f_2}^\dagger(p) a_{f_2 f_1}(p) \\
&- \frac{Kr^2}{a} \sum_{f_1 f_2} \sum_p \left(a_{f_1 f_2}^\dagger(p) a_{f_2 f_1}(p) + a(p)_{f_1 f_2} a_{f_2 f_1}^\dagger(p) \right) \sum_j \cos p_j a \\
&+ \frac{K}{a} \sum_{p f_1 f_2} \left(a_{f_1 f_2}^\dagger(p) a_{f_2 f_1}^\dagger(-p) + a_{f_1 f_2}(-p) a_{f_2 f_1}(p) \right) \sum_j \cos p_j a. \quad (23)
\end{aligned}$$

The Bogoliubov transformation

$$\begin{aligned}
a(p) &\rightarrow a(p) \cosh u(p) + a^\dagger(-p) \sinh u(p), \\
a^\dagger(p) &\rightarrow a^\dagger(p) \cosh u(p) + a(-p) \sinh u(p), \\
a(-p) &\rightarrow a(-p) \cosh u(p) + a^\dagger(p) \sinh u(p), \\
a^\dagger(-p) &\rightarrow a^\dagger(-p) \cosh u(p) + a(p) \sinh u(p), \quad (24)
\end{aligned}$$

will lead to the diagonalization of H_Π , if the parameter u_p satisfies

$$\tanh 2u(p) = -\frac{2G_2}{G_1} \sum_{q=1}^d \cos p_q a, \quad (25)$$

where

$$\begin{aligned}
G_2 &= \frac{K}{a}, \\
G_1 &= 2 \left[M + \frac{Kd}{a}(1-r^2) - \frac{Kr^2}{a} \sum_{q=1}^d \cos p_q a \right]. \quad (26)
\end{aligned}$$

Therefore, after the Bogoliubov transformation, the diagonalized H_{Π} becomes

$$\begin{aligned}
H_{\Pi} &= G_1 \sum_{pf_1f_2} \left(1 - \tanh^2 2u(p)\right)^{\frac{1}{2}} a_{f_1f_2}^{\dagger}(p) a_{f_2f_1}(p) \\
&- \frac{G_1}{2} N_f^2 \sum_p \left[\left(1 - \left(1 - \tanh^2 2u(p)\right)^{\frac{1}{2}}\right) + \frac{2G_2 r^2}{G_1} \sum_{q=1}^d \cos p_q a \right]. \quad (27)
\end{aligned}$$

From (25), (26) and (27), we obtain the dispersion law for the pseudo-scalar mesons, i.e. the relation between the energy E_{Π} of the pseudo-scalar meson and momentum p :

$$\begin{aligned}
E_{\Pi} &= G_1 \left(1 - \tanh^2 2u(p)\right)^{\frac{1}{2}} = G_1 \left[1 - \left(\frac{2G_2 d}{G_1}\right)^2\right]^{\frac{1}{2}} \\
&= 2 \left[M + \frac{Kd}{a}(1 - r^2) - \frac{K(r^2 - 1)}{a} \sum_{q=1}^d \cos p_q a \right]^{\frac{1}{2}} \\
&\times \left[M + \frac{Kd}{a}(1 - r^2) - \frac{K}{a}(r^2 + 1) \sum_{q=1}^d \cos p_q a \right]^{\frac{1}{2}}. \quad (28)
\end{aligned}$$

For naive fermions ($r = 0$) in 3+1 dimensions, according to (28), we reproduce the result of Luo and Chen[5]: $E_{\Pi}^2|_{p_q=0} = m_{\Pi}^2 \propto m$, i.e. the PCAC theorem; In the chiral limit $m \rightarrow 0$, the dispersion law tells us

$$E_{\Pi}^2 = \frac{4K^2}{a^2} \left[9 - \left(\sum_{q=1}^d \cos p_q a\right)^2 \right]. \quad (29)$$

One sees that there exist two zeroes at $\vec{p} = (0, 0, 0)$ and $\vec{p} = (\pi, \pi, \pi)$. This is a well known “doubling problem”. For Wilson fermions ($r \neq 0$), the doubler modes are removed, but the chiral symmetry is explicitly broken. In order to define the chiral limit, one has to fine-tune $M \rightarrow M_c$ so that the pseudo-scalars become massless. From (28), we get

$$M_c = \frac{6Kr^2}{a}. \quad (30)$$

According to this formulae and (28), near the chiral limit, the mass of a pseudo-scalar behaves as

$$\begin{aligned}
m_{\Pi}^2 &= E_{\Pi}^2|_{p_q=0} = 4(M - M_c)^2 + \frac{24K}{a}(M - M_c) \\
&\approx^{M \rightarrow M_c} \frac{24K}{a}(M - M_c), \quad (31)
\end{aligned}$$

which is the PCAC relation for a pseudo-scalar in the Wilson fermion case.

We now consider H_V in (21) similarly. After a Fourier transformation

$$V_j(x) = \sum_p e^{ipx} b_j(p), \quad (32)$$

and a Bogoliubov transformation

$$\begin{aligned} b_j(p) &\rightarrow b_j(p) \cosh v_j(p) + b_j^\dagger(-p) \sinh v_j(p), \\ b_j^\dagger(p) &\rightarrow b_j^\dagger(p) \cosh v_j(p) + b_j(-p) \sinh v_j(p), \end{aligned} \quad (33)$$

where

$$\tanh 2v_j(p) = -\frac{2G_2}{G_1} \left(\sum_{q=1}^d \cos p_q a - 2 \cos p_j a \right), \quad (34)$$

we obtain

$$\begin{aligned} H_V &= \sum_{pjf_1f_2} G_1 \left(1 - \tanh^2 2v_j(p) \right)^{\frac{1}{2}} b_{jf_1f_2}^\dagger(p) b_{jf_2f_1}(p) \\ &- \frac{G_1}{2} N_f^2 \sum_{pj} \left[\left(1 - \left(1 - \tanh^2 2v_j(p) \right)^{\frac{1}{2}} \right) + \frac{2G_2}{G_1} r^2 \sum_{q=1}^d \cos p_q a \right]. \end{aligned} \quad (35)$$

Therefore, using (30), the dispersion relation for the vector mesons is

$$\begin{aligned} E_j &= \frac{2K}{a} \left[3(1+r^2) + (1-r^2) \sum_{q=1}^d \cos p_q a - 2 \cos p_j a \right]^{\frac{1}{2}} \\ &\times \left[3(1+r^2) - (1+r^2) \sum_{q=1}^d \cos p_q a + 2 \cos p_j a \right]^{\frac{1}{2}}. \end{aligned} \quad (36)$$

For naive fermions ($r = 0$) in 3+1 dimensions, from (30) and (36),

$$E_j^2 = \frac{4K^2}{a^2} \left[9 - \left(\sum_{q=1}^d \cos p_q a - 2 \cos p_j a \right) \right], \quad (37)$$

where the chiral limit has been taken. One sees that there exist six zeroes on the boundary of the first Brillouin zone, $\vec{p} = (\pi, 0, 0), (0, \pi, 0), (0, 0, \pi), (\pi, \pi, 0), (0, \pi, \pi), (\pi, 0, \pi)$, due to the “doubling problem”. For Wilson fermions ($r \neq 0$), the doubling problem is avoided, and the mass for a vector meson is

$$\begin{aligned} m_V &= E_j|_{p=0} = G_1 \left(1 - \tanh^2 2v_j(p) \right)^{\frac{1}{2}}|_{p=0} \\ &= G_1 \left[1 - \left(\frac{2G_2}{G_1} (d-2) \right)^2 \right]^{\frac{1}{2}}|_{p=0}, \end{aligned} \quad (38)$$

where (34) has been used. In the chiral limit $M = M_c$,

$$m_V = \frac{4\sqrt{2}}{a}K, \quad (39)$$

which is independent of the Wilson parameter r .

4 Vacuum energy and chiral condensate

The vacuum energy is the vacuum expectation value of the Hamiltonian

$$\begin{aligned} E_\Omega &= \langle \Omega | H | \Omega \rangle = \langle 0 | H_{eff} | 0 \rangle \\ &= (-2N_c N_f N_s) \left(M + \frac{d}{g^2 C_N a} \right) \\ &\quad - \frac{G_1}{2} N_f^2 \sum_p \left[\left(1 - \left(1 - \tanh^2 2u(p) \right)^{\frac{1}{2}} \right) + \frac{2G_2 r^2}{G_1} \sum_{q=1}^d \cos p_q a \right] \\ &\quad - \frac{G_1}{2} N_f^2 \sum_{pj} \left[\left(1 - \left(1 - \tanh^2 2v_j(p) \right)^{\frac{1}{2}} \right) + \frac{2G_2}{G_1} r^2 \sum_{q=1}^d \cos p_q a \right]. \end{aligned} \quad (40)$$

In the $r = 0$ limit, this agrees with Luo and Chen[5].

The fermion condensate

$$\langle \bar{\psi} \psi \rangle = -2N_c N_f N_s + \langle \bar{\psi} \psi \rangle_\Pi + \langle \bar{\psi} \psi \rangle_V \quad (41)$$

can also be computed using the Feynman-Hellmann theorem. Here

$$\begin{aligned} \langle \bar{\psi} \psi \rangle_\Pi &= -\frac{N_f^2}{2} \sum_p \left[\frac{\partial G_1}{\partial m} - \frac{1}{2} \frac{\frac{\partial}{\partial m} (G_1^2 - (2G_2 \sum_{q=1}^d \cos p_q a)^2)}{\sqrt{G_1^2 - (2G_2 \sum_{q=1}^d \cos p_q a)^2}} + \frac{\partial G_2}{\partial m} 2r^2 \sum_{q=1}^d \cos p_q a \right] \\ &= \frac{N_f^2 N_s}{(2\pi)^d} \int_{-\pi/a}^{\pi/a} d^d p \left[\frac{1}{\sqrt{1 - \left[\frac{\frac{K}{a} \sum_{q=1}^d \cos p_q a}{M + \frac{Kd}{a}(1-r^2) - \frac{Kr^2}{a} \sum_{q=1}^d \cos p_q a} \right]^2}} - 1 \right] \\ &\xrightarrow{M \rightarrow M_c} \frac{N_f^2 N_s}{(2\pi)^d} \int_{-\pi/a}^{\pi/a} d^d p \left[\frac{1}{\sqrt{1 - \left[\frac{\frac{K}{a} \sum_{q=1}^d \cos p_q a}{\frac{6Kr^2}{a} + \frac{Kd}{a}(1-r^2) - \frac{Kr^2}{a} \sum_{q=1}^d \cos p_q a} \right]^2}} - 1 \right] \\ &= \frac{N_f^2 N_s}{(2\pi)^d} \int_{-\pi/a}^{\pi/a} d^d p \left[\frac{1}{\sqrt{1 - \left[\frac{\sum_{q=1}^d \cos p_q a}{6r^2 + d(1-r^2) - r^2 \sum_{q=1}^d \cos p_q a} \right]^2}} - 1 \right], \end{aligned}$$

$$\begin{aligned}
\langle \bar{\psi}\psi \rangle_V &= \frac{N_f^2 N_s}{(2\pi)^d} \sum_j \int_{-\pi/a}^{\pi/a} d^d p \left[\frac{1}{\sqrt{1 - \left[\frac{\frac{K}{a} \left(\sum_{q=1}^d \cos p_q a - 2 \cos p_j a \right)}{M + \frac{Kd}{a}(1-r^2) - \frac{Kr^2}{a} \sum_{q=1}^d \cos p_q a} \right]^2}} - 1 \right] \\
&\xrightarrow{M \rightarrow M_c} \frac{N_f^2 N_s}{(2\pi)^d} \sum_j \int_{-\pi/a}^{\pi/a} d^d p \left[\frac{1}{\sqrt{1 - \left[\frac{\sum_{q=1}^d \cos p_q a - 2 \cos p_j a}{6r^2 + d(1-r^2) - r^2 \sum_{q=1}^d \cos p_q a} \right]^2}} - 1 \right]. \quad (42)
\end{aligned}$$

Due to the Wilson term, $\bar{\psi}\psi$ mixes with the identity operator [10]

$$\bar{\psi}\psi^{continuum} = \bar{\psi}\psi^{lattice} + C^I I. \quad (43)$$

This relation is useful for comparing the lattice result with the continuum theory when $1/g^2 \gg 1$, but the computation of the mixing coefficient C^I is quite involved.

5 Discussions and Expectation

In the preceding sections, we have investigated (d+1)-dimensional Hamiltonian LGT with Wilson fermions. Using a strong-coupling perturbative expansion, the effective Hamiltonian in the strong-coupling regime has been obtained and diagonalized exactly by Bogoliubov transformation. Some interesting physical results for the vacuum energy, meson masses and fermion condensate have been obtained. For $r = 0$, our results reduce to those of Luo and Chen[5]. We will apply these techniques to lattice QCD at finite chemical potential.

Acknowledgments

We thank E. Gregory and S.H. Guo for useful discussions. X.Q.L. is supported by the National Science Fund for Distinguished Young Scholars (19825117), Guangdong Natural Science Foundation (990212), Ministry of Education, and Hong Kong Foundation of the Zhongshan University Advanced Research Center. Y.Z.F. is supported partly by the National Natural Science Foundation (19905016).

References

- [1] K.G. Wilson, Phys. Rev. **D10** (1974) 2445; K.G.Wilson, in New Phenomena in Sub-nuclear Physics, Erice Lectures 1975, A. Zichichi, ed. (Plenum, New York, 1977).
- [2] J. Kogut and L. Susskind, Phys. Rev. **D11** (1975) 395.

- [3] J. Smit, Nucl. Phys. **B175** (1980) 307.
- [4] A. Le Yaouance, L. Oliver, O. Pène and J.C. Raynal, Phys. Rev. **D33** (1986) 3098.
- [5] X.Q. Luo and Q. Chen, Phys. Rev. **D46** (1992) 814.
- [6] X.Q. Luo et al., J. Phys. **G16** (1990) 1181; Z. Phys. **C47** (1990)635; Z. Phys. **C51** (1991)423; Phys. Rev. **D50** (1994) 501; Commun. Theor. Phys. **29** (1998) 595; Nucl. Phys. **B(Proc. Suppl.)73** (1999) 942; Phys. Rev. **D59** (1999) 034503; Nucl. Phys. **B(Proc. Suppl.) 63** (1998) 931; Mod. Phys. Lett. **A11** (1996) 2435; Nucl. Phys. **B(Proc. Suppl.)53** (1997) 243.
- [7] Y. Umino, Phys. Lett. **B492** (2000) 385.
- [8] E.B. Gregory, S.H. Guo, H. Kröger and X.Q. Luo, Phys. Rev. **D62** (2000) 054508.
- [9] S.H. Guo, J.M. Liu, Q.Z. Chen, and L. Hu, Commun. Theor. Phys. **3** (1984) 575.
- [10] X.Q. Luo, Z. Phys. **C48** (1990) 283; Commun. Theor. Phys. **16** (1991) 505; Q.Z. Chen and X.Q. Luo, Phys. Rev. **D42** (1990) 1293.
- [11] M. Peskin and D. Schroeder, An Introduction to Quantum Field Theory, Perseus Books, 1995.